

A 2 rebit gate universal for quantum computing

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We show, within the circuit model, how any quantum computation can be efficiently performed using states with only real amplitudes (a result known within the Quantum Turing Machine model). This allows us to identify a 2-qubit (in fact 2-rebit) gate which is universal for quantum computing, although it cannot be used to perform arbitrary unitary transformations.

Performing universal quantum computation is generally equated with the ability to build up arbitrary unitary transformations acting on n qubits, out of a set of unitary transformations that act on a small number of qubits at a time. Deutsch [1] originally presented a single 3 qubit universal quantum gate from which all n qubit unitary transformations could be built. It was subsequently shown that two qubit gates suffice [2].¹

In general, when evaluating a new proposal for implementing quantum computation, the standard procedure is to check whether one can perform (i) a controlled-NOT (CNOT) operation between two qubits, and (ii) arbitrary single qubit unitary transformations. If so, then universal quantum computing is certainly possible. Recently some beautiful ideas for implementing quantum computation by performing measurements on appropriate states have been presented [3]. However the general principle of proving universality in accordance with the ability to obtain evolution corresponding to (i) and (ii) has still been followed.

The ability to perform a CNOT and arbitrary single qubit gates allows one to evolve an n qubit state to any point in the Hilbert space, i.e. it allows the construction of arbitrary unitary transforms. However it has been shown within the Quantum Turing Machine model that universal quantum computing can be performed using only *real* amplitudes [4].

The Quantum Turing Machine model is not particularly intuitive for either thinking about construction of a practical quantum computer, nor for design of quantum algorithms. The purpose of this note is to point out how any quantum computation can be simply translated into a quantum circuit in which all quantum states and gates are real. In particular we will show that the following two qubit gate (written in the computational basis) is

universal for quantum computing:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}, \quad (1)$$

where ϕ is some irrational multiple of π . Note that this gate does *not* allow arbitrary unitary operations to be performed, as can be easily deduced from the fact that the matrix entries are real - and thus it can only produce superpositions of states with real amplitudes. In other words, such a gate cannot in fact evolve us through much, in fact most, of an n qubit hilbert space.

In addition to the ability to perform a gate G , we need the standard assumption that the qubits can be prepared initially in the computational basis. Generally one assumes they can be prepared in the state $|0\rangle$, however we will assume here that they can be prepared in the state $|1\rangle$. This means that by using an ancilla in the state $|1\rangle$ as the control bit, we can implement the single qubit gate $\begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ by implementing G .

The choice of ϕ to be an irrational multiple of π is well known to allow us (by a polynomial number of repeated applications) to efficiently approximate the gate

$$F(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}, \quad (2)$$

for any desired value of θ . Our goal therefore is to show that a gate of the form $F(\theta)$ suffices for universal quantum computation.²

We begin by showing how any quantum computation which uses complex amplitudes, can be replaced by one which is as efficient (in the complexity theoretic sense), but which makes use of only real amplitudes. Imagine the standard quantum algorithm involves the creation at some point of the state:

$$|\psi\rangle = \sum_j r_j e^{i\theta_j} |j\rangle. \quad (3)$$

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¹ While this was surprising from a computer science perspective (because universal reversible classical computing provably requires a three bit gate), it would have been distressing from a physicist's perspective had it been otherwise - all fundamental physical interactions are of a "two-body" form!

² In fact the the use of irrational numbers here is not strictly necessary, following the argument of [4] certain rational numbers will suffice.

If we introduce an ancilla 2-level qubit, the orthonormal states of which we label $|R\rangle$ and $|I\rangle$, then an equivalent state for the purposes of quantum computing is

$$|\psi_E\rangle = \sum_j r_j \cos \theta_j |j\rangle |R\rangle + r_j \sin \theta_j |j\rangle |I\rangle. \quad (4)$$

The purpose of the two-level ‘‘R-I’’ ancilla bit is to keep track of Real and Imaginary parts of the amplitudes which appear in (3). We will use the terminology that the state (4) is the *encoded form* of (3). Note that the probability of obtaining the state $|j\rangle$, upon a measurement in the computational basis, is r_j^2 for both $|\psi\rangle$ and $|\psi_E\rangle$, and of course the amplitudes of $|\psi_E\rangle$ are real.

We need to show that if an efficient algorithm is implemented such that our standard quantum computer now undergoes an evolution

$$|\psi\rangle \rightarrow |\psi'\rangle = \sum_j r'_j e^{i\theta'_j} |j\rangle,$$

then an efficient set of gates $F(\theta)$ can be found such that

$$|\psi_E\rangle \rightarrow |\psi'_E\rangle = \sum_j r'_j \cos \theta'_j |j\rangle |R\rangle + r'_j \sin \theta'_j |j\rangle |I\rangle.$$

We begin by showing how to map arbitrary single qubit gates from the standard to the encoded form, using only $F(\theta)$ gates. In the standard form an arbitrary single qubit rotation can be performed using some combination of the gates

$$R_z(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\tau} \end{pmatrix}, R_y(\tau) = \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix}.$$

Consider the action of $R_z(\tau)$ on a single qubit state in standard form:

$$\begin{aligned} |\psi\rangle &= r_0 e^{i\theta_0} |0\rangle + r_1 e^{i\theta_1} |1\rangle \\ &\rightarrow r_0 e^{i\theta_0} |0\rangle + r_1 e^{i(\theta_1+\tau)} |1\rangle. \end{aligned} \quad (5)$$

The equivalent evolution in encoded form is

$$\begin{aligned} |\psi_E\rangle &= r_0 \cos \theta_0 |0\rangle |R\rangle + r_0 \sin \theta_0 |0\rangle |I\rangle \\ &+ r_1 \cos \theta_1 |1\rangle |R\rangle + r_1 \sin \theta_1 |1\rangle |I\rangle \\ &\rightarrow r_0 \cos \theta_0 |0\rangle |R\rangle + r_0 \sin \theta_0 |0\rangle |I\rangle \\ &+ r_1 \cos(\theta_1 + \tau) |1\rangle |R\rangle + r_1 \sin(\theta_1 + \tau) |1\rangle |I\rangle. \end{aligned} \quad (6)$$

This evolution can be achieved by performing a 2-qubit gate $F(\tau)$, where the *target* qubit is the R-I ancilla. i.e. we control on the first qubit; in the standard encoding this would be the qubit upon which $R_z(\tau)$ acted.

If we now consider the action of $R_y(\tau)$ on a qubit in standard form, it is simple to see that exactly the same evolution can be achieved in encoded form by applying the same gate $R_y(\tau)$ to the equivalent qubit. (Use of the R-I ancilla is not necessary; this gate does not mix real and imaginary parts of amplitudes.) As mentioned previously, this gate can be implemented by $F(\tau)$.

Having seen how to map the action of arbitrary single qubit gates in a standard quantum computation to implementations of $F(\theta)$ on qubits in the encoded form, it remains to be shown that we can implement the encoded form of a non-trivial 2-qubit gate. Generally one chooses a controlled NOT gate, however we will use here the gate $F(\pi/2)$ (which combined with arbitrary single qubit rotations is universal for quantum computing in the standard form). Acting on two qubits in standard form, the gate $F(\pi/2)$ implements

$$\begin{aligned} |\psi\rangle &= r_0 e^{i\theta_0} |00\rangle + r_1 e^{i\theta_1} |01\rangle + r_2 e^{i\theta_2} |10\rangle + r_3 e^{i\theta_3} |11\rangle, \\ &\rightarrow r_0 e^{i\theta_0} |00\rangle + r_1 e^{i\theta_1} |01\rangle + r_2 e^{i\theta_2} |11\rangle - r_3 e^{i\theta_3} |10\rangle. \end{aligned}$$

One can see by inspection that, applying the gate $F(\pi/2)$ in standard form is equivalent to applying it to the corresponding qubits in encoded form - use of the R-I ancilla is again unnecessary.

Thus, gates of the form $F(\theta)$ are universal for quantum computing, and, since they can be efficiently implemented by a gate $G = F(\phi)$ for an appropriately chosen fixed ϕ , we see that the gate G is universal for quantum computing within the circuit model, although it is clearly *not* universal for performing arbitrary unitary transforms. Such considerations may be useful for practical implementations, as well as for probing the more interesting questions about precisely where quantum computers gain their power and to what extent the standard complex Hilbert space formulation of quantum mechanics can actually be argued as necessary as opposed to merely sufficient.

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- [1] D. Deutsch, Proc. R. Soc. London A **425**, 73 (1989).
 - [2] D. DiVincenzo, Phys. Rev. A. **51**, 1015 (1995).
 - [3] R. Raussendorf and H. Briegel, Phys. Rev. Lett., 5188 (2001); M. Nielsen, quant-ph/0108020.

- [4] L. Adleman, J. Demarrais and M-D. A. Huang, Siam. J. Comput. **26**, 1524 (1997).